§ Tangent Panes and Differentials

Recall that, we have defined the tangent plane for $z=f(x,y)$ at $(x_0,y_0)\in D$ to be

$$E=\left\{ \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(y-y_0)+f(x_0,y_0)=z \right\}$$

If we use z_0 to denote $f(x_0,y_0)$, then we have.

$$E=\left\{ \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(y-y_0)-(z-z_0)=0 \right\}$$

We call $\nabla f := \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$ to be the gradient of f

Meanwhile, $\nabla f \perp S_h$ at any $p\in S_h$ (Level curve of f)

Now, we consider a function of 3 variables ; $f(x,y,z)$.

We also have the level surface

$$S_h := \{ (x,y,z) \mid f(x,y,z)=h \}.$$

Again, we have gradient of f

$$\nabla f = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$$

Proposition: $\nabla f \perp S_h$ at every $p\in S_h$

Let $(x_0, y_0, z_0) \in S_{\hbar}$, We have to find the normal vector of the tangent plane at (x_0, y_0, z_0) first.

Idea: Implicit function

$S_{\hbar}$ is a surface, we write $F(x,y,z) = f(x,y,z) - \hbar$

So $S_{\hbar} = \{ F(x,y,z) = 0 \}$

Ideally, we can solve $z = g(x,y)$ with $z_0 = g(x_0, y_0)$

$\Rightarrow$ The normal vector of $\{z = g(x,y)\}$ at (x_0, y_0, z_0)

$$= \left(\frac{\partial g}{\partial x}(x_0, y_0), \frac{\partial g}{\partial y}(x_0, y_0), -1 \right)$$

$$\frac{\partial g}{\partial x} = \frac{\partial z}{\partial x}, \quad \frac{\partial g}{\partial y} = \frac{\partial z}{\partial y}.$$

Here we use the Chain Rule again

$$\begin{cases} x = s \\ y = t \\ z = g(s,t) \end{cases}$$

$$0 = \frac{\partial}{\partial s}\Big(F(s,t,g(s,t))\Big) = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$= \frac{\partial F}{\partial x} \cdot 1 + \frac{\partial F}{\partial y} \cdot 0 + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x}$$

So we have $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$

Similarly, $0 = \frac{\partial}{\partial t}\left(F(s,t,g(s,t))\right) = \frac{\partial F}{\partial x}\cdot 0 + \frac{\partial F}{\partial y}\cdot 1 + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y}$

So $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

Therefore, we have the normal vector of the tangent plane

$$= \left(-\frac{F_x}{F_z}, -\frac{F_y}{F_z}, -1\right)$$

or we can choose

$$(F_x, F_y, F_z) \quad \text{instead.}$$

$F_x = \frac{\partial}{\partial x}F = \frac{\partial}{\partial x}(f-h) = \frac{\partial f}{\partial x}$. Similarly, $F_y = \frac{\partial f}{\partial y}$

$F_z = \frac{\partial f}{\partial z}$. So $(F_x, F_y\ F_z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$= \nabla f \qquad \square$$

Example: Let $f(x,y,z) = x^3 + z^2 + ye^{xz} + z\cos y$

find the tangent plane of S_0 at $(0,0,0)$.

$\nabla f = (3x^2 + yze^{xz}, e^{xz} - z\sin y, 2z + xye^{xz} + \cos y)$

$\nabla f(0,0,0) = (0, 1, 1)$

So $E = \{ y + z = 0 \}$

Suppose $f(x,y)$ is differentiable at $(x_0\ y_0)$

Recall: differentiable at $(x_0\ y_0) \Leftrightarrow$ There exists a linear approximation.

$$\| $$

$$(z - z_0) = \partial_x f(x_0\ y_0)(x - x_0) + \partial_y f(x_0\ y_0)(y - y_0)$$

$$\Rightarrow \Delta z \approx \partial_x f(x_0, y_0)\, \Delta x + \partial_y f(x_0 y_0)\, \Delta y$$

$$\downarrow$$

We write $df = \partial_x f\, dx + \partial_y f\, dy$ to denote this formula

Notice: df, dx, dy are not numbers. They are just some formal notions.

§ Error Bounds

This part will be postponed before we start to learn the Taylor series.

§ Extreme values and Saddle Points :

For one variable function :

$$\text{if} \quad \frac{d^2 f}{dx^2}(x_0) > 0 \Rightarrow f \text{ is convex at } x_0$$

$$\frac{df}{dx}(x_0) = 0 \text{ and } f \text{ is convex at } x_0$$

$$\Rightarrow f(x_0) \text{ is a local min.}$$

For two variable function, we need to check every direction :

$$D_u^2 f(x_0, y_0) > 0 \quad \text{for all } \vec{u}$$

$$D_u f = \partial_x f\, u_1 + \partial_y f\, u_2$$

$$\begin{aligned} D_u^2 f &= D_u(\partial_x f \cdot u_1 + \partial_y f \cdot u_2) \\ &= \partial_x^2 f\, u_1^2 + \partial_y \partial_x f\, u_1 u_2 \\ &\qquad + \partial_x \partial_y f\, u_2 u_1 + \partial_y^2 f\, u_2^2 \\ &= (u_1\ u_2) \begin{pmatrix} \partial_x^2 f & \partial_x \partial_y f \\ \partial_y \partial_x f & \partial_y^2 f \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} \partial_x^2 f & \partial_x \partial_y f \\ \partial_y \partial_x f & \partial_y^2 f \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = A$$

Now, because $f_{xy} = f_{yx}$, so A is a symmetric matrix.

$\Rightarrow A$ is diagonalizable.

, $\exists M$: orthogonal matrix (orthonormal) s.t

$$M^T A M = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{for some } \lambda_1, \lambda_2 \in \mathbb{R}$$

$D_u^2 f(x_0 y_0) > 0$

$$\Leftrightarrow (u_1 u_2) A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} > 0$$

$$\Leftrightarrow (u_1 u_2) \cdot M \cdot M^T A \cdot M \cdot M^T \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} > 0$$

$$\Leftrightarrow (v_1 v_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} > 0$$

$$\Leftrightarrow \lambda_1 v_1^2 + \lambda_2 v_2^2 > 0 \quad \text{for all } (v_1 v_2)$$

So $\lambda_1 > 0$ and $\lambda_2 > 0$.

Here $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = M^T \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ can be regarded as changing of coordinate

Define: $\text{Hess}(f) := \det(A) = \lambda_1 \cdot \lambda_2$

$$= \left(f_{xx} f_{yy} - f_{xy}^2\right)$$

- If $\text{Hess}(f) > 0$, then λ_1, λ_2 have same sign (both possitive or both negative).

In this case, $D_u^2 f(x_0 y_0) > 0$ for all $\vec{u}$

or $D_u^2 f(x_0 y_0) < 0$ for all $\vec{u}$.

So, once we can determine the sign of $D_u^2 f(x_0 y_0)$ for some u, the function f will be convex or concave at $(x_0 y_0)$ respectly.

Prop 1. Suppose that $\text{Hess}(f)(x_0 y_0) > 0$ and

1) $f_{xx}(x_0 y_0) > 0 \Rightarrow f$ is convex at $(x_0 y_0)$

2) $f_{xx}(x_0 y_0) < 0 \Rightarrow f$ is concave at $(x_0 y_0)$

Now, suppose $\text{Hess}(f) < 0$, then we have $\lambda_1 \lambda_2$ have different sign. So for example, $\lambda_1 > 0$, $\lambda_2 < 0$. then we can take

$$\vec{u}_1 = M\begin{pmatrix}1\\0\end{pmatrix}, \quad \vec{u}_2 = M^T\begin{pmatrix}0\\1\end{pmatrix}$$

$$\left.\begin{array}{l} D^2_{\vec{u}_1} f(x_0, y_0) = \lambda_1 > 0 \\ D^2_{\vec{u}_2} f(x_0, y_0) = \lambda_2 < 0 \end{array}\right\} \Rightarrow f \text{ behaves like a saddle pt.}$$

So in addition, we have the following conclusion:

- if $\text{Hess}(f)(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$

  $\Rightarrow f$ is convex at (x_0, y_0)

Moreover, $\nabla f(x_0, y_0) = (0, 0)$ and f is convex at (x_0, y_0)

$\Rightarrow f(x_0, y_0)$ is a local min

- if $\text{Hess}(f)(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$

  $\Rightarrow f$ is concave at (x_0, y_0)

  $\nabla f(x_0, y_0) = (0, 0)$ and f is concave at (x_0, y_0)

  $\Rightarrow f(x_0, y_0)$ is a local max

- if $\text{Hess}(f)(x_0, y_0) < 0$ and $\nabla f(x_0, y_0) = (0, 0)$

  $\Rightarrow f$ has a saddle point at (x_0, y_0)